

An Introduction to the Fourier Transform: Relationship to MRI

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OBJECTIVE. The Fourier transform, a fundamental mathematic tool widely used in signal analysis, is ubiquitous in radiology and integral to modern MR image formation. Understanding MRI techniques requires a basic understanding of what the Fourier transform accomplishes. MR image encoding, filling of k-space, and a wide spectrum of artifacts are all rooted in the Fourier transform.

CONCLUSION. This article illustrates these basic Fourier principles and their relationship to MRI.

Joseph Fourier (1768–1830) is credited with observing that a complex signal can be rewritten as the infinite sum of simple sinusoidal waves [1]. Fourier himself is most famous for applying this principle to solving an array of differential equations governing heat dissipation. However, applications of this concept are invaluable to anyone who wishes to study the composition of a complex signal, whether it is in the form of music, voices, images, or digital medical imaging, including MRI.

Any complicated signal or wave can be rewritten as the sum of a series of simple waves. An approximation of a complicated wave can be achieved by adding together very simple sine and cosine waves (Fig. 1) with varying combinations of frequencies and amplitudes (Fig. 2). The Fourier series (Fig. 3) provides a means to describe a complicated wave in terms of simple sines and cosines. The more of them we add together, each with successively higher frequency, the better the approximation (Fig. 4).

The Fourier transform decomposes a complicated signal into the frequencies and relative amplitudes of its simple component waves.

The Fourier transform (Fig. 5A) allows us to study the frequency content of a variety of complicated signals [1]. We can view and even manipulate such information in a Fourier or frequency space (Fig. 5B). A familiar clinical example of a Fourier space is the frequency and amplitude spectrum obtained

during MR spectroscopy (Fig. 5C). The multiple peaks (amplitudes) in the MR spectroscopy spectrum represent the relative contributions of particular biologic metabolites in a given region of interest (ROI), each differing slightly in resonance frequency as a result of its unique chemical structure [2].

MR Image Encoding and Filling k-Space

With regard to MRI, the complicated signal we wish to decompose is the MR echo containing the frequency- and phase-encoded spatial information necessary to construct an image. Following the slice selection gradient for a typical spin-echo sequence, which isolates a particular imaging plane, all spin systems precess at the same frequency and phase as dictated by the main magnetic field (B_0). At this point, all protons in the desired imaging plane look the same to the Fourier transform. The application of superimposed, dynamically changing gradient fields introduces spatially dependent variations in frequency and phase across the ROI, effectively interrogating the anatomy for all different spatial frequencies. The steeper the applied gradient, the greater degree of achievable separation of spin systems is possible. Strong gradients are necessary to seek out high spatial frequencies (detail), whereas less steep gradients bring out lower spatial frequencies (contrast). In addition, these smaller gradients generate much more collective signal than steep gradients because precessional frequencies and phase are overall more similar

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across the ROI (this will be evident when we compare the central portions of k-space with the periphery, discussed later). The amplitude of the returning echoes will vary with tissue composition, TR, and TE [3, 4].

This otherwise hopelessly complicated signal is digitized, dismantled by the Fourier transform, and entered into k-space, a 2D Fourier space that organizes spatial frequency and amplitude information (Fig. 6). One pixel in k-space, when inverse-transformed, contributes a single, specific spatial frequency (alternating light and dark lines) to the entire image. A 2D inverse Fourier transform of the entirety of k-space combines all spatial frequencies, and results in the image we see. Depending on where a pixel resides in k-space, the lines will be of varying frequency and orientation. By convention, high spatial frequencies are mapped to the periphery of k-space and low spatial frequencies are mapped near the origin. The relative intensity of a pixel reflects its overall contribution to the image, with brighter pixels contributing more of a particular spatial frequency.

Radiofrequency spike is an artifact that exemplifies this concept quite nicely. A spark or other source of radiofrequency noise in the MR scanner room can contaminate the MR echo. When Fourier-transformed, the frequency of that spark may be erroneously incorporated into k-space as an abnormal, bright pixel. The rogue frequency is then inverse-Fourier-transformed into sinusoidal noise in the image space (Figs. 7A and 7B). Zipper artifact is another manifestation of radiofrequency leak. It is caused by a constant, narrow range of radiofrequency emission that is occasionally emanating from patient monitoring devices. When this unwanted signal is inverse-Fourier-transformed into the image space, it manifests as persistent, thin, hyperintense lines in the frequency-encoding direction that are thought to resemble a zipper [3] (Fig. 7C).

Depending on how and when we choose to activate a particular combination of phase- and frequency-encoding gradients, we have the option of filling k-space in several creative ways. If we inspect the Fourier space of a photograph of Lincoln (Fig. 8A), the most intense portions (brightest pixels) are located centrally where low frequencies reside (contrast). This is where the most essential components of the image are stored. By corollary, when filling k-space, we may choose to fill the central, high-signal-to-noise portions and ignore the less important, low-signal-to-noise regions

to reduce acquisition time. This can be accomplished by activating gradients corresponding to the center of k-space, perhaps in a spiral fashion [3] (Fig. 8B). Furthermore, the symmetric organization of k-space, a direct consequence of complex conjugate symmetry properties inherent to the Fourier transform, has been used to decrease acquisition time by acquiring only half of k-space [3] (Fig. 8C). A mirror image of the remaining half can then be generated, saving time at the expense of the signal-to-noise ratio.

Phase-encoding involves quickly activating, then deactivating a gradient. While the gradient is transiently on, some spin systems will precess faster than others, depending on their location along the gradient. When turned off, the rate of precession across the ROI will equilibrate; however, a spatially dependent, gradual change in phase will have been “imprinted” on the protons [3, 4]. Successively increasing the phase-encoding gradient amplitude will create a varying rate-of-change of phase across the ROI. This rate-of-change of phase translates into a kind of frequency that the Fourier transform resolves into different spatial frequencies [5] (Fig. 9A). The greater number of phase-encoding steps performed, the greater the resulting spatial resolution (Fig. 9B).

Artifacts

Gibbs artifact is an imperfect approximation of sharp edges by a Fourier series lacking an adequate number of high-frequency terms. This effect is easily shown by removing high spatial frequencies from the Fourier space of an image of Lincoln and inverse-transforming the result (Figs. 10A and 10B). In MRI, this is commonly referred to as truncation or ringing artifact, and it becomes noticeable when too few phase-encoding steps are performed. It is often seen near the inner table of the calvarium (Fig. 10C) or in the upper cervical spinal cord, where it can be mistaken for a syrinx. Increasing the number of phase-encoding steps (e.g., from 128 to 256) will ameliorate this artifact.

Because it consumes the most time in signal acquisition, motion artifact or ghosting most noticeably occurs in the phase-encoding direction. In the time it takes to apply a new phase-encoding amplitude step (approximately seconds), a moving structure may have assumed a new position and thus a new resonance frequency. Phase-encoding an abrupt change in position is essentially approximating a sharp edge (frequency shift)

with sinusoidal waves (phase-encoding), leading to ringing artifact in its Fourier series [6] (Fig. 11). When these phase-encoding errors are inverse-Fourier-transformed, the structure appears to be spread out over the image in the phase-encoding direction, regardless of the original direction of motion.

Aliasing or wraparound artifact (Fig. 12) is also related to phase-encoding and Fourier misregistration. To understand this artifact, recall that a phase shift of exactly 2π radians or 360° will superimpose two waves exactly, and thus negate any benefit of imparting spatial variation based on phase. This leaves $-\pi$ to $+\pi$ (-180° to $+180^\circ$) available to phase-encode a given field of view. Aliasing artifact occurs when excited spin-systems from outside the field of view (less than $-\pi$ or more than $+\pi$) overlap with those of identical phases inside the field of view. Mathematically indistinguishable, these structures are assigned by the Fourier transform to the same spatial position in the image space, causing them to wrap around to the other side [3].

Chemical shift (India ink artifact) is a spatial misregistration phenomenon occurring in the frequency-encoding direction. Protons in fat and water precess at slightly different resonance frequencies (the gap becoming more prominent with increasing main magnetic field strength), with fat precessing slower than water by about 3.5 ppm. During frequency-encoding, signal from a single voxel containing fat and water is assigned two discrete spatial positions based on these two resonance frequencies [3] (Fig. 13A). The result is an accentuated bright or dark margin corresponding to fat-water interfaces. If resonance frequencies from fat and water are not resolved as different, additive fat and water signals from a given voxel will result in oscillating peaks and troughs depending on the TE, forming the basis for in-phase and out-of-phase imaging (Fig. 13B).

Summary

The Fourier transform is a fundamental tool in the decomposition of a complicated signal, allowing us to see clearly the frequency and amplitude components hidden within. In the process of generating an MR image, the Fourier transform resolves the frequency- and phase-encoded MR signals that compose k-space. The 2D inverse Fourier transform of k-space is the MR image we see. A grasp of the Fourier transform is essential to understanding several MR artifacts and the myriad of methods of signal acquisition in practice today.

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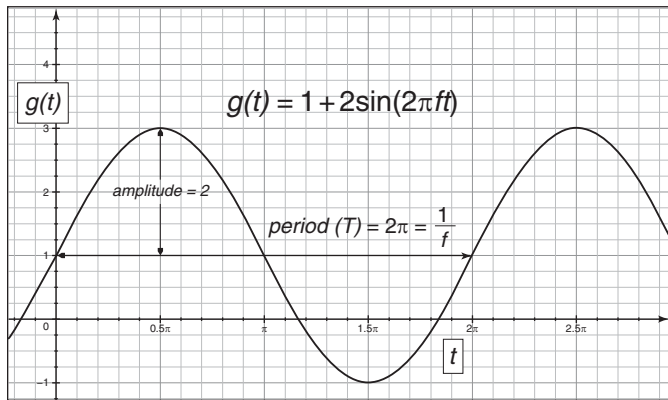


Fig. 1—Adding simple waves. Example sine wave shown here, $1 + 2\sin(2\pi ft)$, has frequency $f = 1 / (2\pi)$, period $T = 2\pi$, amplitude = 2, and is centered at $g(t) = 1$.

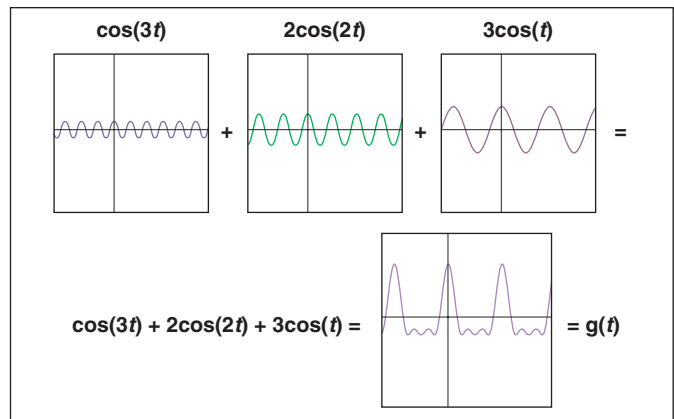


Fig. 2—Complicated waves. A complicated wave $g(t)$ can be obtained by adding together simpler waves.

The average value of $g(t)$ over one period T centers this wave vertically in space.

$$a_0 = \frac{2}{T} \int_0^T g(t) dt$$

Multiplying $g(t)$ by the simple cosine or sine wave in question, integrating the result, and dividing by half the period leaves an expression for a specific amplitude (a_n or b_n) while everything else cancels out.

$$a_n = \frac{2}{T} \int_0^T g(t) \cos(2\pi nft) dt$$

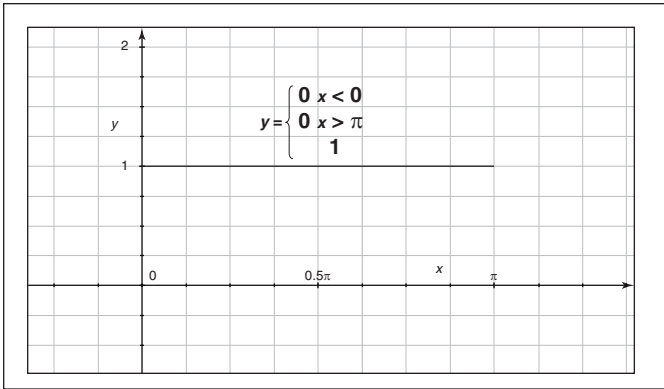
$$b_n = \frac{2}{T} \int_0^T g(t) \sin(2\pi nft) dt$$

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi nft) + b_n \sin(2\pi nft)$$

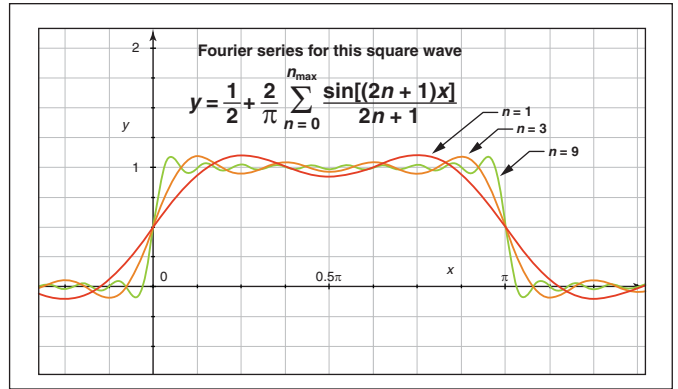
A complicated signal $g(t)$ = The infinite sum of simple cosine and sine waves of varying amplitudes (a_n and b_n) and progressively increasing frequency f .

Fig. 3—Fourier series for $g(t)$. A complicated wave $g(t)$ can be rewritten as an infinite sum of simple cosine and sine waves by progressively increasing their fundamental frequency f by integers n , and by varying their amplitudes, a_n and b_n . If we substitute $g(t) = 1$ (a square wave) into the equations shown here, we obtain expressions for a_0 , a_n , and b_n that can be inserted into the Fourier series. After simplifying, we are left with a Fourier approximation for a square wave.

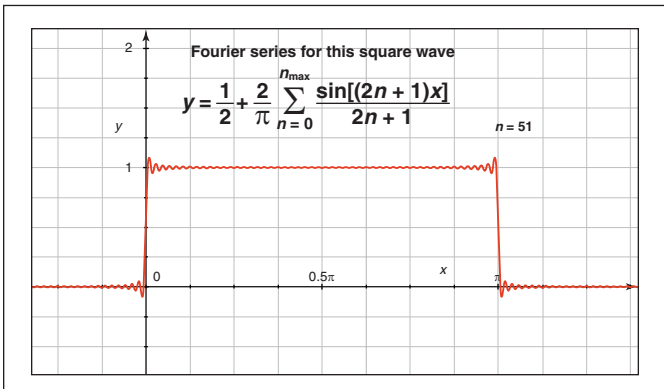
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A



B

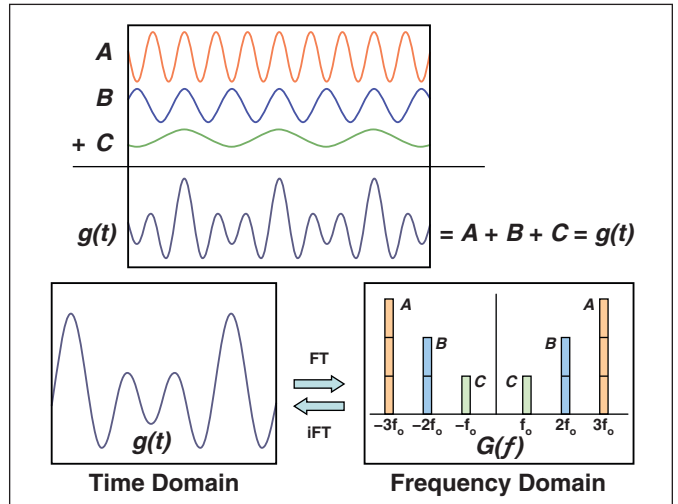


C

Fig. 4—Fourier series for a square wave.
A, A square wave, given by y (equal to 1 for $0 < x < \pi$ and equal to 0 everywhere else), can be approximated with increasing accuracy by addition of simple sine and cosine waves of progressively increasing frequency.
B and C, In waves shown in **B**, $n = 1, 3, 9$; in **C**, $n = 51$.

Fourier Transform: $G(f) = \int_{-\infty}^{\infty} g(t) e^{-i2\pi ft} dt$ **Inverse Fourier Transform:** $g(t) = \int_{-\infty}^{\infty} G(f) e^{i2\pi ft} df$

A



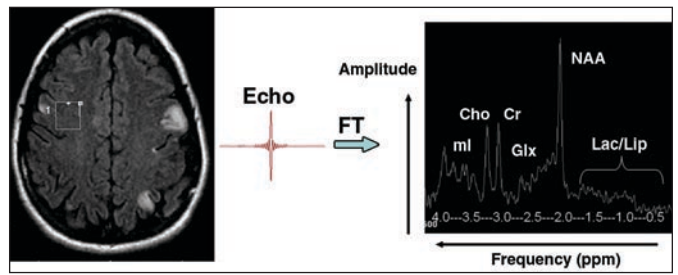
B

Fig. 5—Fourier transform (FT).

A, Fourier transform of a complicated signal $g(t)$, which exists in time (t) or spatial domain, gives an expression for frequency domain $G(f)$. When plotted, frequency domain displays individual frequencies and relative amplitudes of simpler waves constituting $g(t)$. Inverse Fourier transform (iFT) of $G(f)$ restores the time domain. No information is gained or lost in mathematic transforms; they merely change the way we see the same information.

B, Fourier transform (FT) extracts the frequencies and relative amplitudes of the simpler waves hidden in a complicated wave $g(t)$. Inverse Fourier transform (iFT) restores the time domain. In this example, Fourier transform of three cosine waves of different frequencies results in three delta functions.

C, MR spectroscopy. In contrast to MRI, which uses resonance frequencies and phase to encode an image, MR spectroscopy addresses a smaller region of interest (ROI) with a specific radiofrequency pulse bandwidth. Multiple neuronal metabolites (ml, myoinositol; Cho, choline; Cr, creatine; Glx, glutamate and glutamine; NAA, *N*-acetyl aspartate; Lac, lactate; Lip, lipid) resonate at characteristic frequencies on the basis of their unique chemical structure. The returning MR spectroscopy echo is a composite signal of many different echoes from metabolites in the ROI, which is resolved into individual resonance frequencies and their relative amplitudes (abundance) by the Fourier transform. The term "relative" is an important qualifier because the Fourier transform cannot measure the absolute nature of any frequency. The height of a peak in the MR spectroscopy Fourier spectrum makes sense only relative to another peak.



C

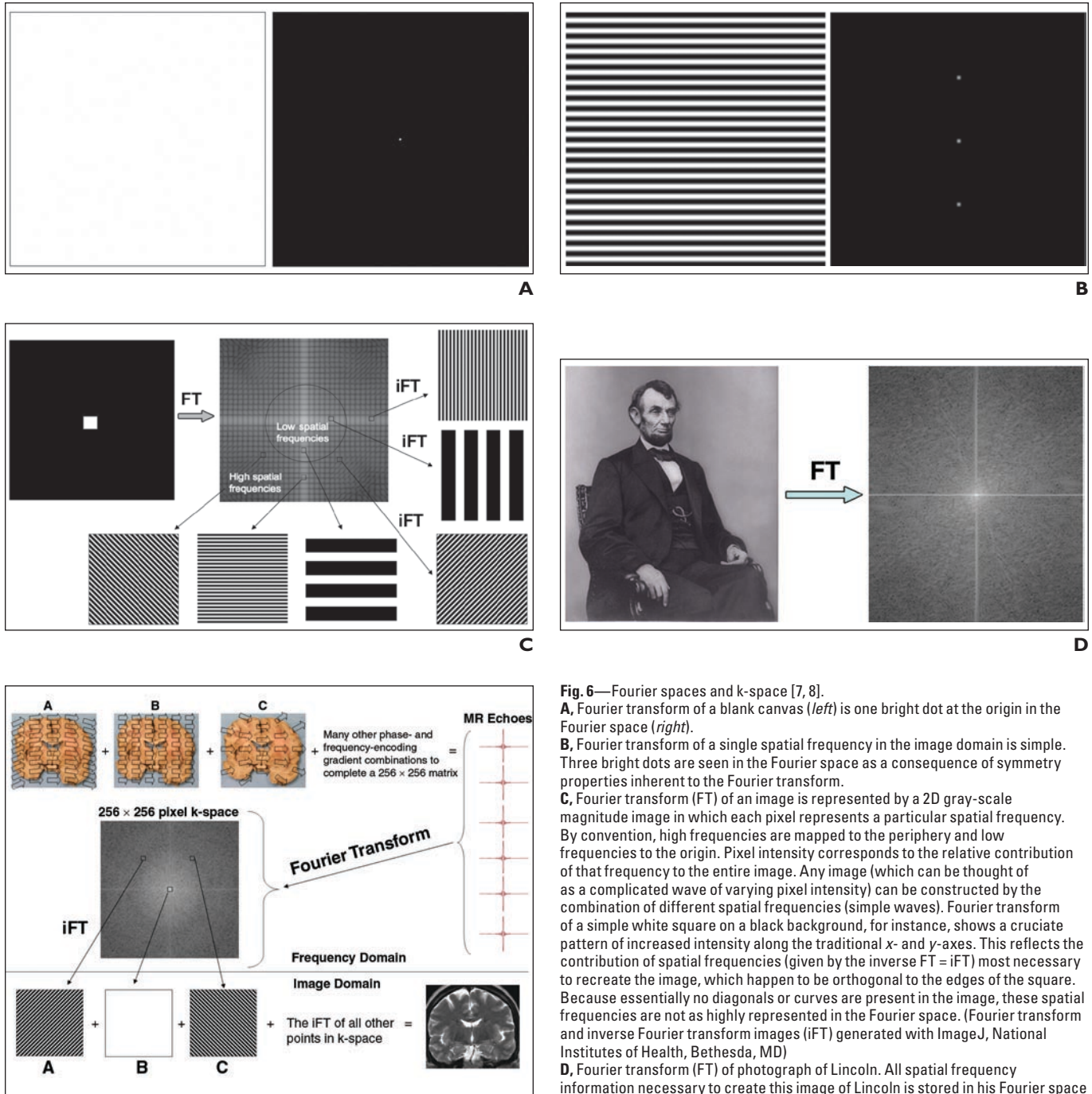


Fig. 6—Fourier spaces and k-space [7, 8].
A, Fourier transform of a blank canvas (*left*) is one bright dot at the origin in the Fourier space (*right*).
B, Fourier transform of a single spatial frequency in the image domain is simple. Three bright dots are seen in the Fourier space as a consequence of symmetry properties inherent to the Fourier transform.
C, Fourier transform (FT) of an image is represented by a 2D gray-scale magnitude image in which each pixel represents a particular spatial frequency. By convention, high frequencies are mapped to the periphery and low frequencies to the origin. Pixel intensity corresponds to the relative contribution of that frequency to the entire image. Any image (which can be thought of as a complicated wave of varying pixel intensity) can be constructed by the combination of different spatial frequencies (simple waves). Fourier transform of a simple white square on a black background, for instance, shows a cruciate pattern of increased intensity along the traditional *x*- and *y*-axes. This reflects the contribution of spatial frequencies (given by the inverse FT = iFT) most necessary to recreate the image, which happen to be orthogonal to the edges of the square. Because essentially no diagonals or curves are present in the image, these spatial frequencies are not as highly represented in the Fourier space. (Fourier transform and inverse Fourier transform images (iFT) generated with ImageJ, National Institutes of Health, Bethesda, MD)
D, Fourier transform (FT) of photograph of Lincoln. All spatial frequency information necessary to create this image of Lincoln is stored in his Fourier space (*right*). As discussed previously, a single pixel in the image does not have a single pixel correlate in the Fourier space. Rather, each pixel in Fourier space contributes a spatial frequency to the overall image of Lincoln.
E, MRI. This coronal slice of a brain is interrogated for all its different spatial frequencies by successively altering magnetic field gradients (*open arrows in top three images*) during frequency- and phase-encoding. Although only three examples are shown here, many different gradient combinations are necessary to fill k-space. Inverse Fourier transform (iFT) of k-space essentially adds the relative contributions of all spatial frequencies to give the final image.

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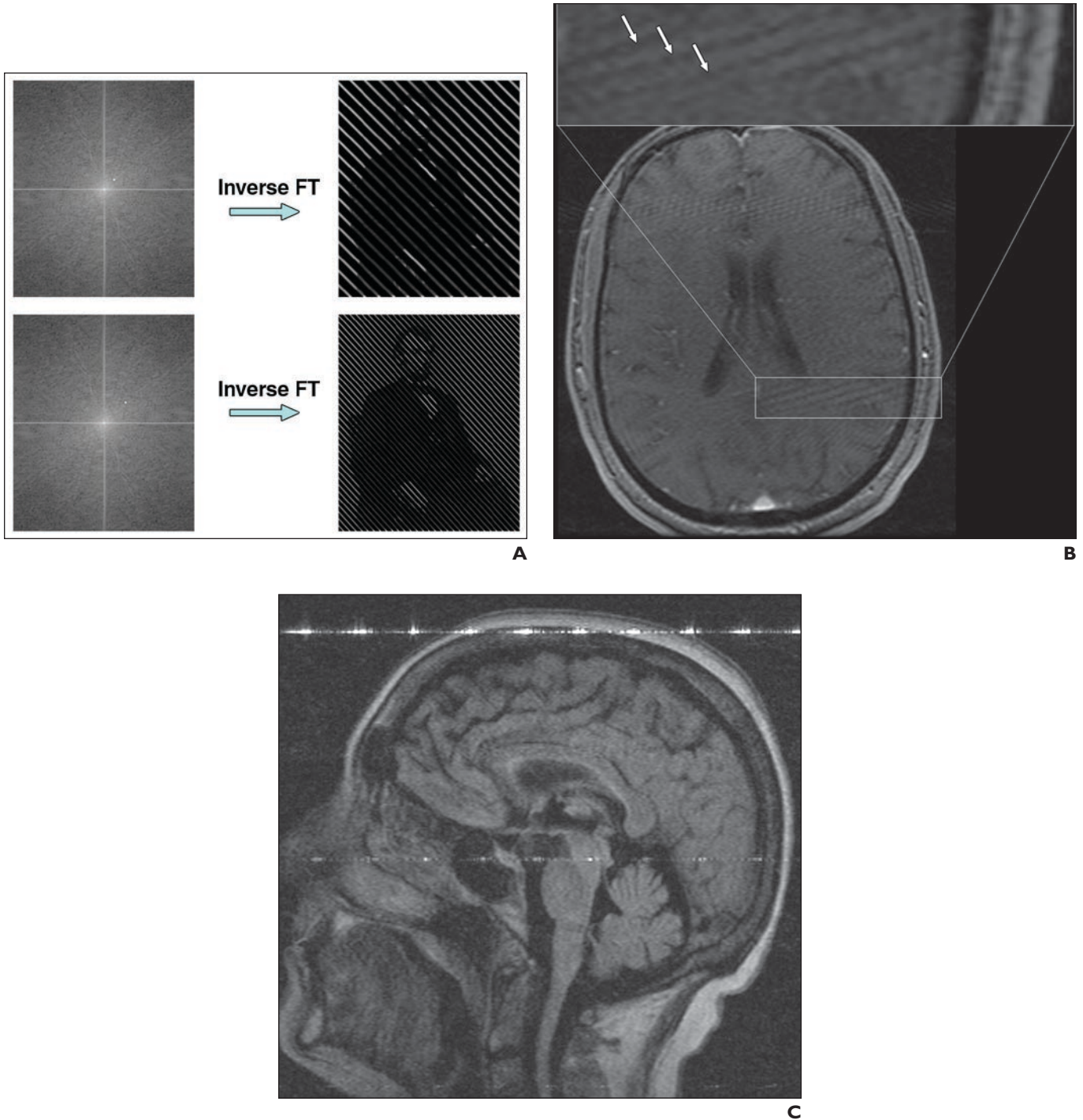
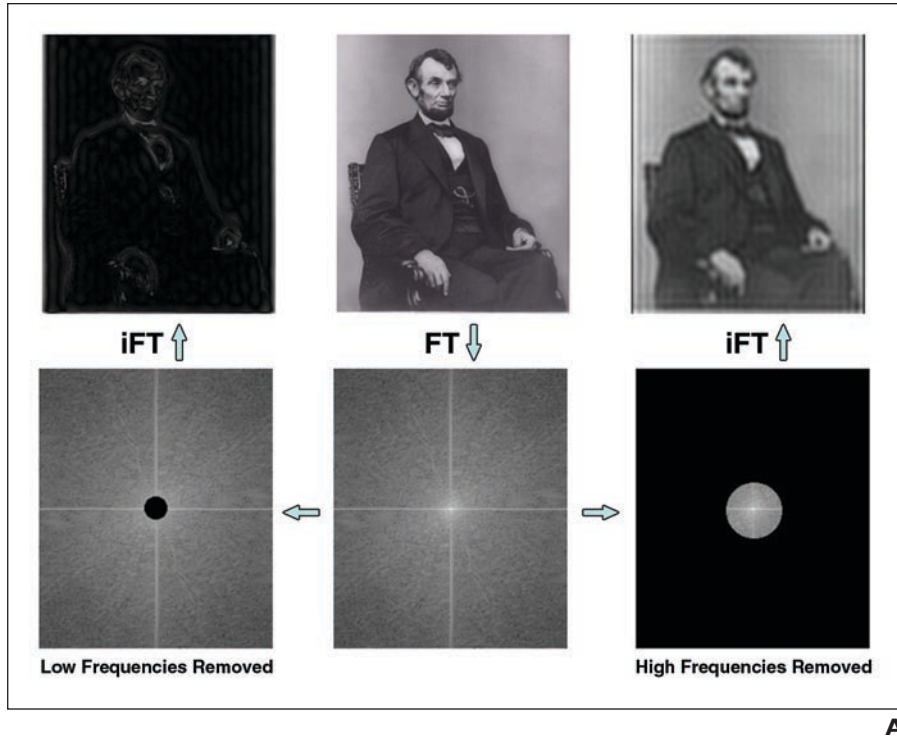


Fig. 7—Radiofrequency spike artifact [7, 8].

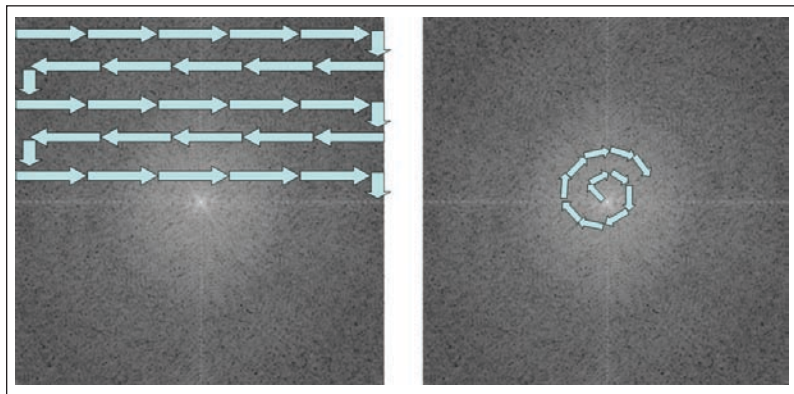
A, One abnormal, bright pixel in a Fourier space is transformed into sinusoidal noise in the image space. If moved slightly farther away from the origin, spatial frequency is higher. A spark in the MR scanner room, erroneously integrated into k-space, may result in radiofrequency spike artifact. FT = Fourier transform.

B, Door to MR scanner was left open just a crack during this acquisition. Notice regular pattern of striations (*arrows*) present in image, a result of radiofrequency leak.

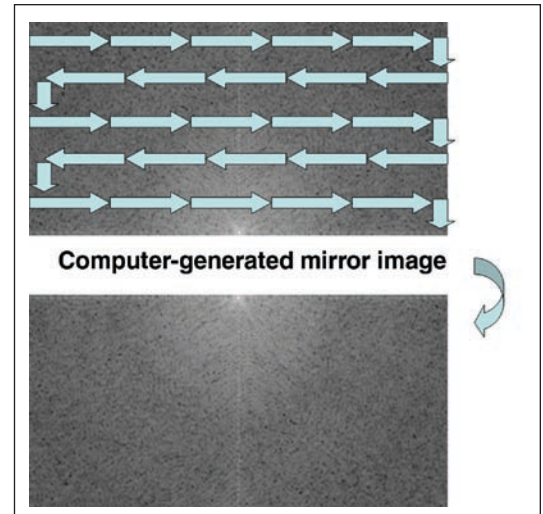
C, "Zipper" artifact from radiofrequency leak. During this sagittal FLAIR acquisition, radiofrequency noise from a patient monitor is transformed into intense thin bright lines through the image. Artifact reflects a narrow range of contaminant frequencies manifest in frequency-encoding direction. Every image during this acquisition was degraded by the same intense lines in the same location.



A



B



C

Fig. 8—Differential filling of k-space.

A, When low frequencies are removed from Fourier space of Lincoln (*upper left*), sharp edges are preserved in image at the expense of contrast resolution. When high frequencies are removed, image contrast is preserved; however, it is blurry and demonstrates Gibbs artifacts (see Fig. 10). Observe how few spatial frequencies are actually necessary to recreate a recognizable image of Lincoln. FT = Fourier transform, iFT = inverse Fourier transform.

B, By steering frequency- and phase-encoding gradients appropriately during MR image acquisition, k-space can be filled not only sequentially line by line, but also in a spiral fashion about the origin. Filling the essential, high-signal-to-noise, central portions of k-space can save considerable time and result in a recognizable image. This comes at the expense of fine detail, which is stored in the periphery of k-space (as depicted in **A**).

C, Fourier transform formula makes use of exponentials of imaginary numbers (e^i) to represent simple waves, and as a result the Fourier transform yields both real and imaginary information displaying complex conjugate symmetry. Half-Fourier techniques exploit this symmetry by acquiring only half of k-space and generating a mirror image of the remaining half. Such a time-saving mechanism comes at the expense of signal-to-noise, however, because only half of the potential signal is actually acquired.

Fourier Transform and MRI

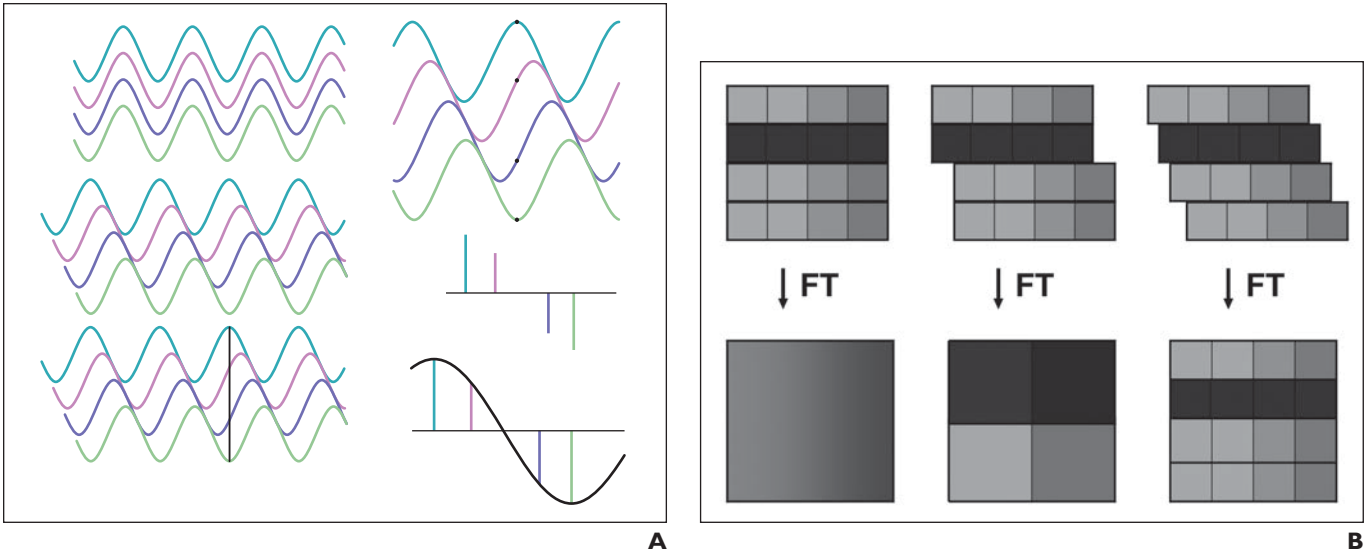


Fig. 9—Phase-encoding.

A, Spin systems are purposely dephased across a region of interest to create spatial variation in phase-encoding direction. The four cosine waves in this figure are shifted slightly out of phase. If a line intersects the middle of the waves, and the changing amplitude along this line is plotted, it corresponds to its own wave. This rate-of-change of phase corresponds to a frequency that Fourier transform can resolve. Each phase-encoding step is performed at different gradient amplitudes, resulting in differing degrees of phase change.

B, Columns in 4×4 matrix image each correspond to a specific frequency, depending on location (frequency-encoded). Without phase-encoding (*left*), Fourier transform (FT) cannot resolve any differences in brightness in vertical direction because all frequencies are identical and all amplitudes (*brightness*) are blurred together. The addition of a phase shift (*middle*, implied by a shift in the boxes to the right) imparts uniqueness to the boxes in the vertical (phase-encoding) direction so brightness is partially resolved. The greater the number of phase-encoding steps, the better the resolution (*right*) [8].

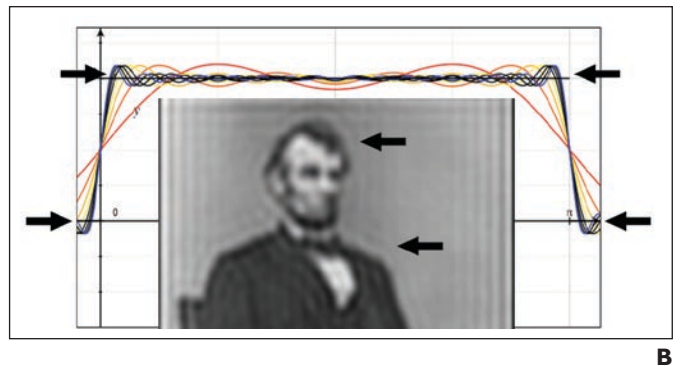
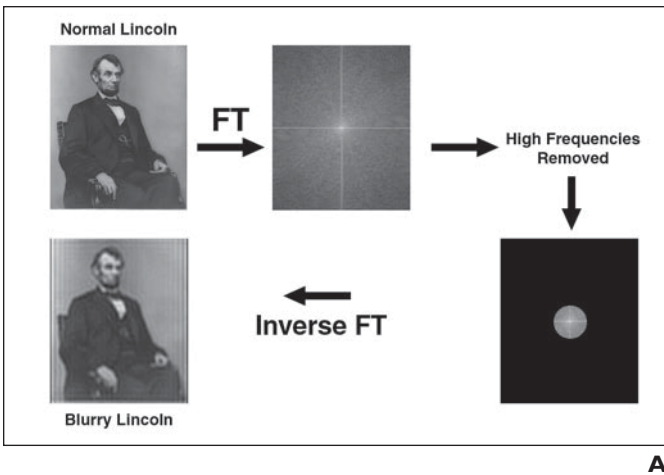
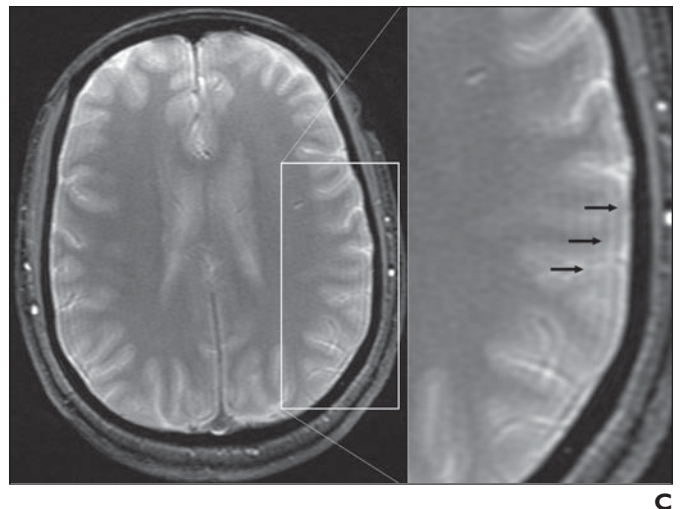


Fig. 10—Gibbs artifact (also called truncation or ringing artifact).

A, High spatial frequencies were removed from this image of Lincoln. When inverse-transformed, not enough frequencies are available to approximate sharp edges, resulting in Gibbs artifact and blurring.

B, Gibbs phenomenon is evident mathematically and in manipulated image of Lincoln (*arrows*).

C, Axial gradient-echo image of brain obtained at 256×160 matrix. Gibbs artifact near inner table of calvarium manifests as subtle hypointense lines overlying cortex (*arrows*).



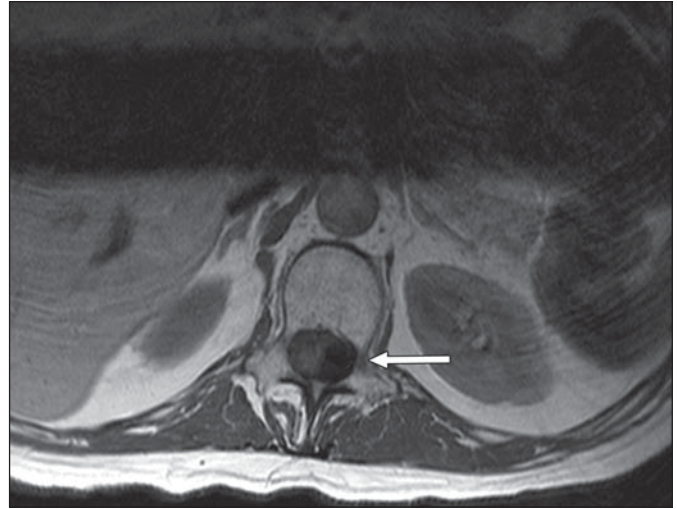
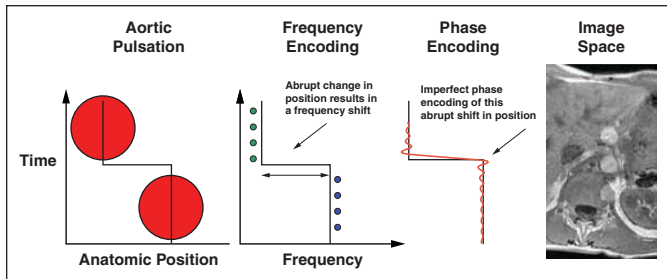


Fig. 11—Motion.

A, Ghosting (motion). An abrupt change in the position of a structure results in a shift along the frequency-encoding gradient and a change in precessional frequency. Phase-encoding an abrupt shift in position is similar to approximating a sharp edge with a Fourier series. Ripples in its Fourier series propagate in the phase-encoding direction. For a structure with periodic motion such as aortic pulsation, these errors are incorporated into k-space in a periodic fashion, resulting in duplicates of the moving structure propagating in the phase-encoding direction, regardless of the direction of the original motion.

B, In this axial T1-weighted MR image, pulsation artifact from aorta simulates a hypointense epidural lesion (arrow). Swapping frequency- and phase-encoding directions can often redirect this artifact away from target anatomy.

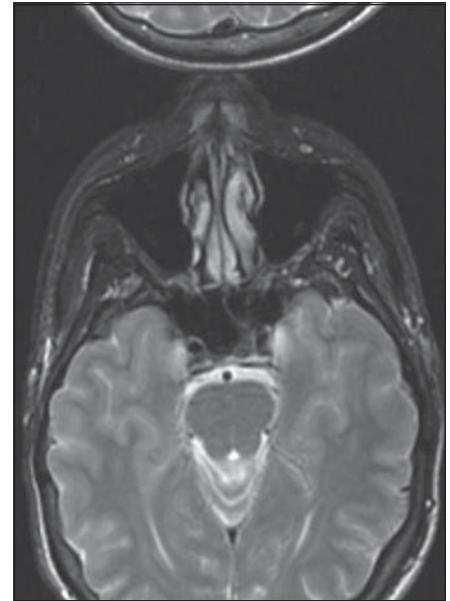
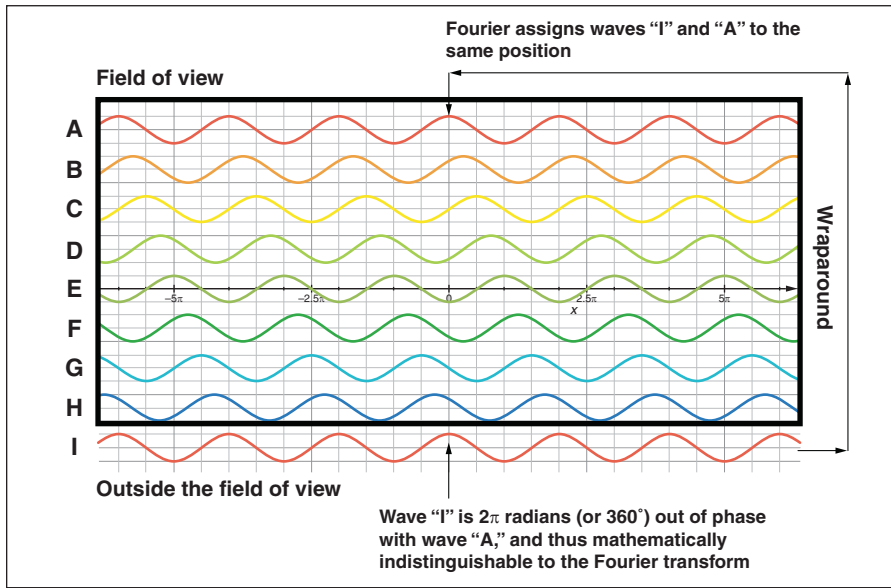
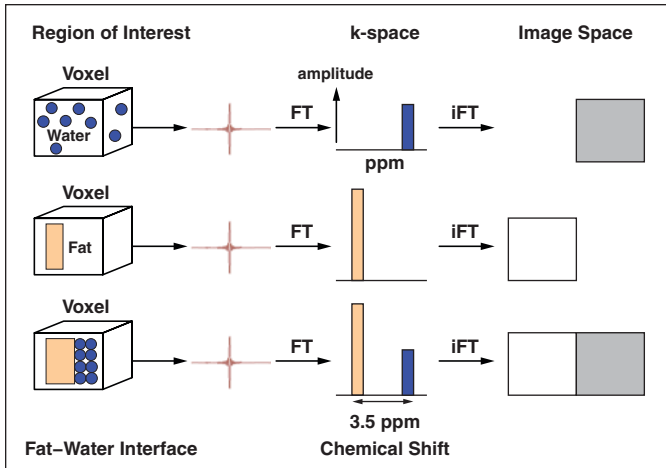


Fig. 12—Wraparound.

A, Wraparound (aliasing). Only phase shifts between 2π radians or 360° are available to encode an image. The phase shift in this image (depicted by waves A–H) covers the field of view (black rectangular outline). Just outside the field of view, wave "I" has assumed a phase shift of 2π radians (360°) and is mathematically identical to wave "A" on the opposite side of the field of view. Fourier transform assigns structures encoded by "I" to positions encoded by "A," giving the wraparound phenomenon.

B, Axial T2-weighted image shows back of head (excluded from field of view) wrapping around to the front.

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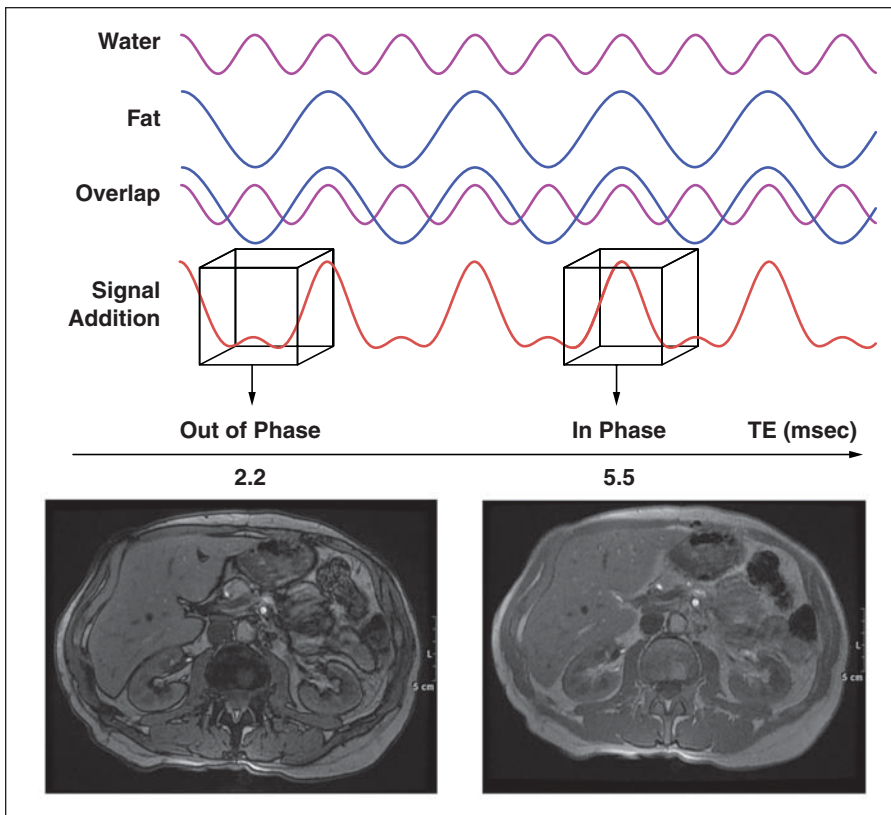


A

Fig. 13—Chemical shift.

A, Chemical shift artifact occurs when a voxel in the body contains both fat and water. When signal from such a voxel is Fourier-transformed, peaks corresponding to both fat and water (each differing in amplitude, depending on TR and TE) resonate at slightly different frequencies, separated by 3.5 ppm at 1.5 T). Fourier transform (FT) assigns two separate spatial locations to a single voxel on the basis of these different frequencies (chemical shift), despite their common origin. iFT = inverse Fourier transform.

B, Axial gradient-echo out-of-phase (*left*, TR/TE, 150/2.236) and in-phase (*right*, 150/5.516) images through abdomen. When fat (lower frequency) and water (higher frequency) signals from a single voxel are added, alternating peaks and troughs occur at regular time intervals. At TE of 2.236 (*left image*), a sharp, dark margin delineating fat–water interfaces (around liver, kidneys, muscles, and so forth) represents signal trough from voxels sharing water and fat. At TE of 5.516 (*right image*), the restored signal replaces the sharp interface.



B

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